Group Theory 10

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Second Isomorphism theorem:-

Let N and T be subgroups of G with $N \trianglelefteq G$. Then $N \cap T$ is normal in T and $T/(N\cap T) \cong NT/N$

$$\begin{tikzcd}
 & NT \arrow[dl, dash] \arrow[dr, dash] & \\
N \arrow[dr, dash] & & T \arrow[dl, dash] \\
 & N\cap T &
\end{tikzcd}$$

Proof:-

Third Isomorphism Theorem:-

Let $K \leq H \leq G$ where both K and H are normal subgroups of G. Then H/K is a normal subgroup of G/K and $(G/K)/(H/K) \cong G/H$

Q) Prove that a homomorphism $f: G \to H$ is an injection if and only if $\ker f = 1$

Ans:- Let f be injective.

$x \in \ker f \quad f(x) = 1 = f(e) \Rightarrow x = e$

Let $\ker f = 1$,

$f(a) = f(b) \Rightarrow f(ab^{-1}) = 1 \quad \Rightarrow ab^{-1} \in \ker f$

$\Rightarrow ab^{-1} = e$

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$$\Rightarrow ab^{-1} = e$$
$$\Rightarrow a = b$$

Q) Let $N \triangleleft G$ and let $f : G \to H$ be homomorphism whose kernel contains N. Show that f induces a homomorphism

$$f_* : G/N \longrightarrow H \quad \text{by} \quad f_*(Na) = f(a)$$

Ans:- $\ker f \supset N$

$$G/N = \{ aN : \cdot \}$$

$$f(N) = 1 \qquad f(Na) = f(a)$$

$$f_* : G/N \to H$$
$$f_*(N) = 1$$
$$f_*(Na) = f(a) = f_*(N) f_*(a)$$

Direct Products:-

Def:- If H and K are groups then their direct product denoted by $H \times K$ is the group with elements (h, k) where $h \in H$ and $k \in K$ with operation,

$$(h, k)(h', k') = (hh', kk')$$

$(1, 1) \in H \times K$

$(h, k)^{-1} = (h^{-1}, k^{-1}) \in H \times K$

Q) Show that $H \times K$ is abelian iff H and K are abelian.

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Q> If p is a prime then prove that $\mathbb{Z}/p^2 \not\cong \mathbb{Z}/p \times \mathbb{Z}/p$

Ans:-

$p^2-1 \in \mathbb{Z}/p^2 \quad \text{Ord}(p^2-1) = p^2$ But no element of $\mathbb{Z}p \times \mathbb{Z}/p$ satisfy it.

Q> Define an isomorphism between $\mathbb{Z}^*_{p^2}$ and $\mathbb{Z}/p \times \mathbb{Z}/p-1$

Ans:- $|\mathbb{Z}^*_{p^2}| = \phi(p^2) = p^2 - p = p(p-1)$

$|\mathbb{Z}/p \times \mathbb{Z}/p-1| = p(p-1)$

$\mathbb{Z}/p$ is cyclic

$f : \mathbb{Z}/^*_{p^2} \longrightarrow \mathbb{Z}/^*_p$

$\hookrightarrow$ surjective map

$\text{Im} f \in \mathbb{Z}/^*_p \Rightarrow$ so $\text{Im} f$ has order $p-1$

$\hookrightarrow \cong \mathbb{Z}/p-1$

$\text{Ker} f = \{ a : f(a) = 1 \} = 1 + p\mathbb{Z}$

$|\text{Ker} f| = p$

$\text{Ker} f \cong \mathbb{Z}/p$

$\mathbb{Z}^*_{p^2} / \mathbb{Z}/p \cong \mathbb{Z}/p-1$

$\mathbb{Z}/p^2 \cong \mathbb{Z}p \times \mathbb{Z}p-1$

Q> Let $H = \{a \in \mathbb{Q} : a = 3^n 8^m$ for some n and m in $\mathbb{Z}\}$

Prove that H under multiplication is subgroup of $\mathbb{Q} \setminus \{0\}$

Ans:— $1 \in H$ and $3^0 8^0 \in H$

$a, b \in H$, $\quad a = 3^{n_1} 8^{m_1}$

$\qquad b = 3^{n_2} 8^{m_2}$

$$ab^{-1} = 3^{n_1 - n_2} 8^{m_1 - m_2} \in H$$

Q> Let D be the set of all elements of finite order in an Abelian group G. Prove that D is a subgroup of G.

Ans:— $1 \in D$ $\qquad \left.\begin{array}{l} a^n = 1 \\ b^m = 1 \end{array}\right\}$ for some n, m finite

$a, b \in D$

$$(ab^{-1})^{mn} = a^{mn} b^{-mn} = 1$$

so, $ab^{-1} \in D$

Q> G is of order $2^2 \times 13$. x, y are two distinct elements of G of order 2. Then H is generated by x, y

Ans:— $x^2 = 1 \quad y^2 = 1$

$1, x, y \in H$ $\qquad 1, x, y, xy \in H$

$\qquad 1, x, y, yx, xy \in H$

$xy \neq x^2 \Rightarrow xy \neq 1, x, y$

$\quad \neq y^2$

$\quad \neq 1$

$G = \mathbb{Z}_{52}, \ \mathbb{Z}_4 \times \mathbb{Z}_{13}, \ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}$

Q> Let a, b be elements in a group such that $ab = ba$ and $Ord(a) = 25$ and $Ord(b) = 49$ Prove that G contains an element of order 35

Ans :- $ab = ba$ $\quad gcd(25, 49) = 1$

$Ord(ab) = lcm(25, 49) = 25 \times 49$

Let $x = (ab)^{35}$

$x^{35} = (ab)^{25 \times 49} = 1$

$Ord(x) = 35$ $\quad ab \in G$

$(ab)^{35} \in G \Rightarrow x \in G$